Analysis I Lecture 12

Some mistake from last time

Definition Liminf Limsup

Let (x_n) be a bounded sequence

define $y_n = \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}$

$z_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}$

then $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} z_n$

Proposition Let (x_n) be bounded then

$\liminf_{n \to \infty} (x_n)$ and $\limsup_{n \to \infty} (x_n)$ exist.

Proof liminf exists:

Sequence (y_n) is monotone and bounded therefore by monotone converges theorem (y_n) converges

Monotonicity:

$$y_{n+1} = \inf \underbrace{\{x_{n+1}, x_{n+2}, \dots\}}_{A}$$

$$\rotatebox{90}{$\geq$}$$

$$y_n = \inf \underbrace{\{x_n, x_{n+1}, x_{n+2}, \dots\}}_{B}$$

$\hookrightarrow$ $y_n \leq y_{n+1}$ and (y_n) is increasing.

$$\boxed{\begin{array}{c} A \subset B \\ \inf B \leq \inf(A) \end{array}}$$

Boundedness:

(x_n) is bounded above that is

$\exists\ C$ s.t. $\forall n \in \mathbb{N}$ $x_n < C$

But therefore $y_n = \inf\{x_n, x_{n+1} \dots\}$

satisfies $y_n \leq x_n < C$

$\Rightarrow$ (y_n) is bounded above.

So by monotone convergence theorem:

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} y_n \text{ exists.}$$

- The proof for $\limsup_{n\to\infty} x_n$ is the same $\blacksquare$

Proposition A sequence (x_n) converges if and only if

1) (x_n) is bounded

2) $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$

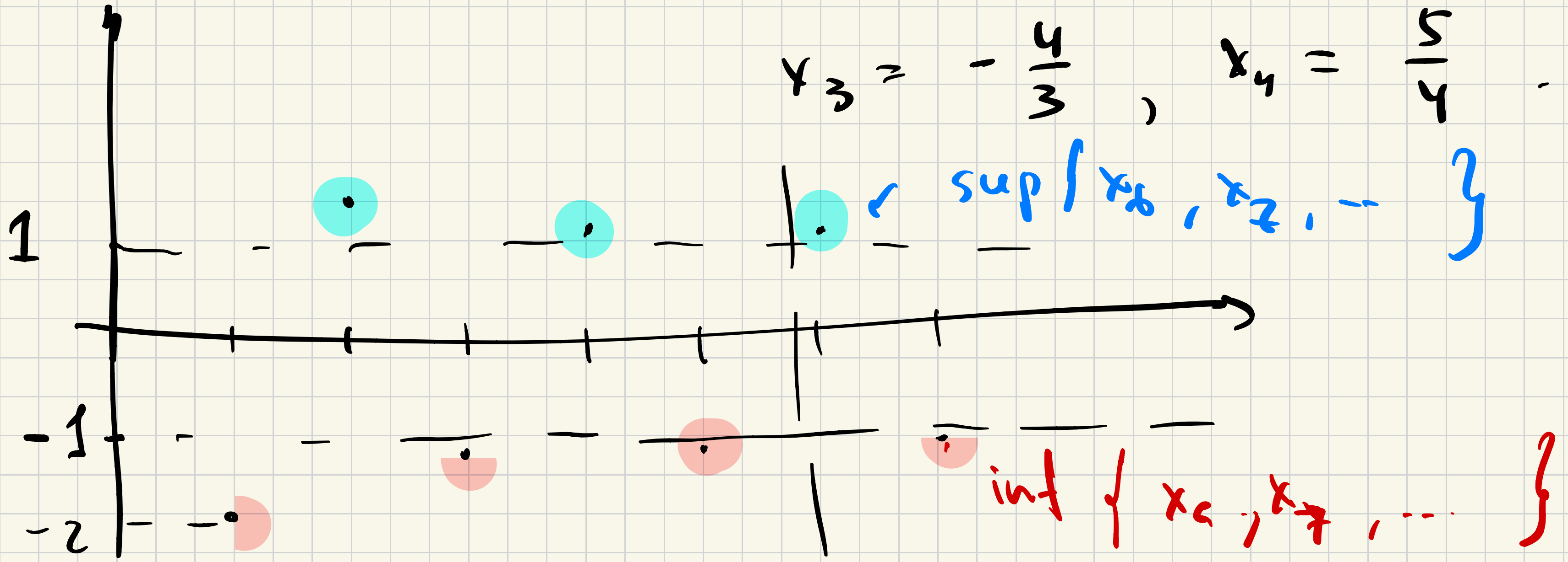
And in this case $\lim x_n = \liminf x_n = \limsup x_n$.

Example

$x_n = (-1)^n \cdot \left(1 + \frac{1}{n}\right) \quad n \geq 1$

$x_1 = -2 \qquad x_2 = (-1)^2 \cdot \left(1+\frac{1}{2}\right) = 1{,}5$

$x_3 = -\frac{4}{3}, \quad x_4 = \frac{5}{4} \quad \ldots$



$\limsup_{n\to\infty} x_n = 1 \qquad \liminf_{n\to\infty} x_n = -1$

Today: Series

How to make sense of infinite sums?

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = ?$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = ?$$

Idea: Think about infinite sums as limits of finite sums (called partial sums or truncated sums)

I want to evaluate

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \ldots$$

I can look at $\displaystyle\sum_{n=0}^{K} \frac{1}{2^n}$ for $k \to \infty$

And then I can take a limit of $\left(\displaystyle\sum_{n=0}^{K} \frac{1}{2^n}\right)$.

Definition Let (a_n) be a sequence then the series of (a_n) is a sequence (S_n) defined by

partial sums

$$S_n = \sum_{i=0}^{n} a_i$$

truncated sums

If (S_n) converges we denote by $\sum_{i=0}^{+\infty} a_n$ the $\lim_{n\to\infty} S_n$

If (s_n) converges we denote by $\sum_{i=0}^{\infty} a_i$ the

$\lim_{n\to\infty} s_n$

- In this case we say that series of a_n converges.

- Sometimes we will write $\sum_{n=0}^{+\infty} a_n$ for the sequence of partial sums.

Definition A series $\sum_{i=0}^{\infty} a_i$ converges absolutely if the series

$$\sum_{i=0}^{\infty} |a_i|$$

converges.

Example Geometric series

Let $a_n = \frac{1}{2^n}$ then

$$s_n = \sum_{i=0}^{n} \frac{1}{2^i} = \frac{1-\frac{1}{2^{n+1}}}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^{n+1}}\right)$$

$\to 0$

So $\lim_{n\to\infty} s_n = 2 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$

In general: $a_n = q^n$

to show this

$(1+q+\dots+q^n)\cdot(1-q) = 1-q^{n+1}$

$\Rightarrow S_n = 1 + q + q^2 + \dots + q^n = \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q}\left(1 - q^{n+1}\right)$

$\Rightarrow$ If $|q| < 1$ then $\lim_{n\to\infty} q^{n+1} = 0$

$\Rightarrow$ (S_n) converges and $\boxed{\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}}$

Example Harmonic series

$a_n = \frac{1}{n}$, $s_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$

We will see that (s_n) approaches $+\infty$

so $\boxed{\sum_{n=1}^{\infty} \frac{1}{n} = +\infty}$

Example Another definition of e

Recall: $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ exists

by monotone convergence thm.

We define e to be this limit.

Def e is the limit of $\left(1+\frac{1}{n}\right)^n$.

Alternatively: $a_n = \frac{1}{n!}$

$$\Rightarrow s_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$$

$\lim_{n\to\infty} s_n = \sum_{n=0}^{\infty} \frac{1}{n!}$ exists and $\boxed{\sum_{n=0}^{\infty} \frac{1}{n!} = e}$

In other words $\sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$.

Example   For $\alpha \geq 0$

$$\sum_{k=1}^{+\infty} \frac{1}{k^\alpha}$$

converge for $\alpha > 1$

diverge for $\alpha \leq 1$

Proposition

- If $\sum_{k=0}^{+\infty} a_k$ exists then $\lim_{k\to\infty} a_k = 0$

Proof idea: Assume $\lim_{k\to\infty} a_k \neq 0$ or does not exists

We get contradiction with Cauchy's criterion for s_n.

Recall: Cauchy's criterion: If (x_n) converges iff

$$\forall \varepsilon > 0 \quad \exists N \quad \text{s.t.} \quad \forall n, m > N$$

$$|x_n - x_m| < \varepsilon$$

But:

$$|s_{n+1} - s_n| = \left| \sum_{i=0}^{n+1} a_i - \sum_{i=0}^{n} a_i \right| =$$

$$= |a_{n+1}|$$ this gives a contradiction $\blacksquare$

Example

$a_n = c \neq 0 \Rightarrow \sum_{n=0}^{\infty} a_n$ diverges

$a_n = (-1)^n \Rightarrow \sum_{n=0}^{\infty} a_n$ diverges

$\vdots$

Convergence criteria

Theorem If series converges absolutely, it converges.

$$\sum_{n=0}^{+\infty} |a_n| \text{ converges} \Rightarrow \sum_{n=0}^{+\infty} a_n \text{ converges}$$

Corollary

Let (a_n) any sequence with

- $a_n \geq 0 \quad \forall n \in \mathbb{N}$
- $\sum_{n=0}^{+\infty} a_n$ converges then

any sequence obtained from (a_n) by introducing signs arbitrarily also converges.

E.g. $\quad a_n = \frac{1}{2^n}$

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & a_4 & \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \end{array}$$

$$\begin{array}{cccccc} -1 & -\frac{1}{2} & +\frac{1}{4} & -\frac{1}{8} & +\frac{1}{16} & \dots \\ b_0 & b_1 & b_2 & \dots & & \end{array}$$

$|b_i| = |a_i| = a_i \Rightarrow \sum_{n=0}^{+\infty} b_n$ converges absolutly

$\Rightarrow \sum_{n=0}^{+\infty} b_n$ conv.

Theorem (Squeeze theorem for series)

Let $\sum_{n=0}^{+\infty} a_n$, $\sum_{n=0}^{+\infty} b_n$ be two series

1) If $\sum_{n=0}^{\infty} b_n$ converges and $\exists K \in \mathbb{N}$ s.t.

$\forall k \geq K$ $|a_k| \leq b_k$ then

$\sum_{k=0}^{+\infty} a_k$ converges absolutely

(and thus converges)

(ii) If $\sum_{k=0}^{+\infty} b_k = +\infty$ and $\exists K \in \mathbb{N}$

$\forall k > K \quad 0 \leq b_k \leq a_k$ then $\sum_{k=0}^{+\infty} a_k = +\infty$.

Example   divergence of harmonic series.

$$1 \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8}$$

Idea is to construct sequence b_n

with $b_n \leq \frac{1}{n}$ and s.t. $\sum_{n=0}^{\infty} b_n = +\infty$

$$1 \quad \underbrace{\frac{1}{2} \quad \frac{1}{3}}_{2 \text{ elements}} \quad \underbrace{\frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7}}_{4 \text{ elements}} \quad \underbrace{\frac{1}{8} \quad - - -}_{8 \text{ elements}}$$

$$\frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{16} \quad \frac{1}{16} \quad - - -$$

$$\underbrace{b_1}_{1/2} \quad \underbrace{b_2 \quad b_3}_{\frac{1}{4}+\frac{1}{4}=\frac{1}{2}} \quad \underbrace{b_4 \quad - \quad - \quad b_7}_{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}} \quad \underbrace{b_8 \quad - - - \quad b_{15}}_{\frac{1}{2}}$$

Notice that

- $b_i < \frac{1}{i}$
- $\sum_{i=2^n}^{2^{n+1}-1} b_i = \frac{1}{2}$

$$\sum_{i=2^n}^{2^{n+1}-1} b_i = \frac{1}{2} \implies$$

$$\implies S_{2^n-1} = n \cdot \frac{1}{2} = \frac{n}{2}$$

$$S_3 = b_1 + b_2 + b_3 = \frac{1}{2} + \frac{1}{2} = 2 \cdot \frac{1}{2} = 1$$

$$2^{2}-1$$

$\implies (S_n)$ approaches $+\infty$ and $\sum_{n=0}^{\infty} b_n = +\infty$.

By squeeze theorem $\left(\sum_{n=0}^{\infty} b_n = +\infty \text{ or } b_n < \frac{1}{n} \right)$

We get that

harmonic series diverge.

$\blacksquare$

<u>Example</u> Convergence of $\sum_{n=0}^{\infty} \frac{1}{n!}$.

Idea is to construct b_n s.t.

$$b_n \geq \frac{1}{n!} \quad \text{and} \quad \sum_{n=0}^{\infty} b_n \text{ converges}$$

You can take $b_n = \frac{1}{2^{n-1}}$

Notice that $\quad 2^{n-1} \leq n!$

$$2^0 = 1 \leq 1! = 1$$

$$2^1 \leq 2!$$

$$2 \cdot 2 = 2^2 < 3! = 2 \cdot 3$$

$$2^n = \underbrace{2 \times 2 \times \dots 2}_{n \text{ times}} < 2 \times 3 \times 4 \times \dots (n+1)$$

So $\frac{1}{2^{n-1}} \geq \frac{1}{n!}$ But

on the other hand

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$ exists so

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$ also exists.